# Combining Loop and Fock Quantizations for Cosmological Universes with Perturbations

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# The model

- We consider **perturbed** FRW universes filled with a **massive** scalar field.
- The scalar field is minimally coupled.
- The model can generate inflation.

- The most interesting case is flat spatial topology. It is also the simplest.
- The effects of **spatial curvature** can be studied by considering, e.g., spherical topology.
- We assume **compact** spatial sections.

# The model

It's been well studied, even in LQC, though...

- Anomalies: not a starting point for quantization.
- Effective dynamics: Needs a true derivation.

- **Approximations**: As few as possible. Should be derived or at least checked for consistency.
- In many cases these checks are only internal, within the approximated description.

# Perturbations about flat FRW

- Truncation at **<u>quadratic</u>** order in the action.
- Includes **<u>backreaction</u>** at that order.
- Tests the validity of less refined truncations and allows one to develop **approximation** methods, controlling their range of application.

# Hybrid approach

Effects of quantum geometry are only accounted for in the background

- Succesfully applied in Gowdy cosmologies.
- In those cases there is no truncation.
- In the present case, we only deal with the quadratically perturbed model.

# Uniqueness of the Fock description

- **Ambiguity** in selecting a Fock representation in QFT in curved space-times.
- This can be restricted by appealing to *background* **symmetries**.
- Typically this is not sufficent in non-stationarity.
- Proposal: demand the **UNITARITY** of the quantum evolution.

  The conventional interpretation of QM is guaranteed (beyond the viewpoint of algebraic quantizations).

- There is a natural ambiguity in the **separation of the background** from the field.

  In cosmology, this introduces time-dependent canonical field transformations.

- Remarkably, symmetry invariance and dynamical unitarity select a **UNIQUE canonical pair** and a UNIQUE Fock representation for their CCR's.

# Uniqueness of the Fock description

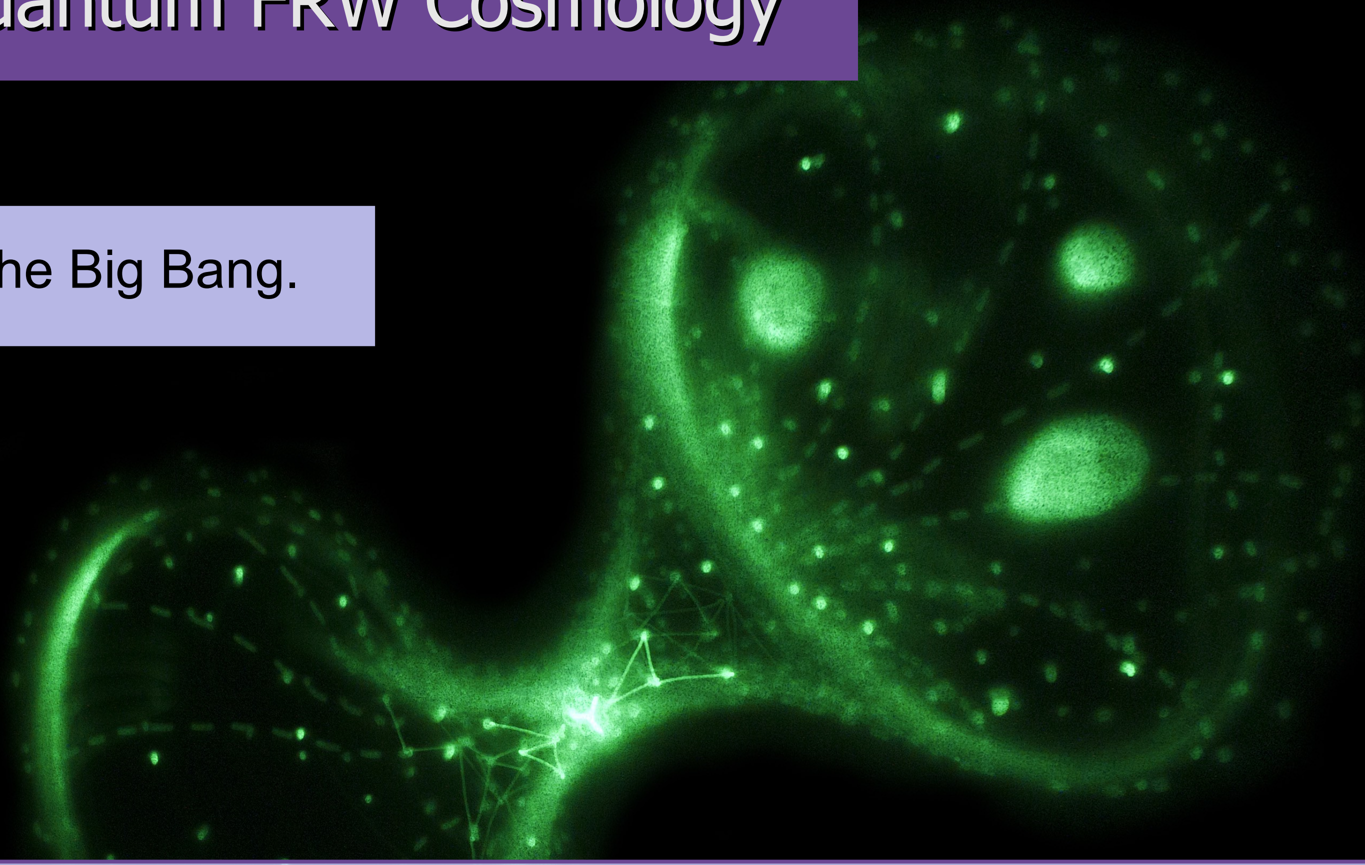
# Uniqueness of the Fock description

- Recent works DO NOT incorporate the correct scaling (AA&N). This affects the quantum description, and in particular the *effective* approaches therein dereived.

- Moreover, one can even consider non-local canonical transformations, respecting the decoupling of field modes.

The **UNIQUENESS** of the quantization, up to unitary equivalence, is guaranteed.

# Loop Quantum FRW Cosmology



- Avoids the Big Bang.

- Specific **proposal** such that:
  - Evolution can be defined even without ideal clocks (masless field).
  - The WdW limit is unambiguous in each superselection sector.
  - It is optimal for numerical computation.

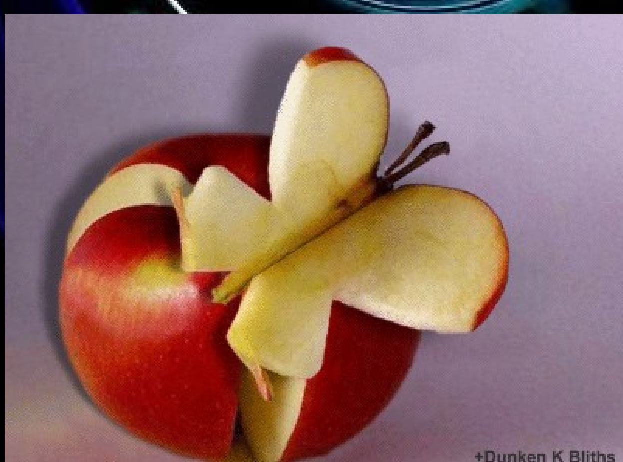
# Classical system: FRW

- **Massive scalar field minimally coupled to a compact, flat FRW universe.**

**Geometry:** $A^i_a = c\,{}^0e^i_a\,(2\pi)^{-1}; \quad E^a_i = p\sqrt{{}^0e}\,{}^0e^a_i\,(2\pi)^{-2}.$ $\quad \{c, p\} = 8\pi G\gamma/3.$

$$\Rightarrow \quad a^2 = e^{2\alpha} = |p|(2\pi\sigma)^{-2}; \quad \pi_\alpha = -pc(\gamma\, 8\pi^3\sigma^2)^{-1}. \qquad \sigma^2 = G(6\pi^2)^{-1}.$$

**Matter:** $\varphi = (2\pi)^{3/2}\sigma\,\phi; \quad \pi_\varphi = (2\pi)^{-3/2}\sigma^{-1}\pi_\phi.$




$$V = |p|^{3/2}.$$

**Hamiltonian constraint:**

$$C_0 = -\frac{6}{\gamma^2}\sqrt{|p|}\,c^2 + \frac{8\pi G}{V}\left(\pi_\phi^2 + m^2 V^2 \phi^2\right).$$

# Classical system: Modes

$\vec{n}\in Z^3$, $n_1\geq 0$.

- We expand inhomogeneities in a (real) **Fourier basis**:

$$Q_{\vec{n},+}=\frac{1}{2\pi^{3/2}}\cos\vec{n}\cdot\vec{\theta}\,,\qquad Q_{\vec{n},-}=\frac{1}{2\pi^{3/2}}\sin\vec{n}\cdot\vec{\theta}\,.$$

- The basis is **orthonormal**, and we exclude the zero mode in the expansions.

- These functions are eigenmodes of the Laplace-Beltrami operator of the standard flat metric on the three-torus, with eigenvalue $-\omega_n^2=-\vec{n}\cdot\vec{n}\,.$

- We only consider **scalar perturbations**: decoupled from vector and tensor perturbations at dominant order.

# Classical system: Inhomogeneities

- Mode expansion of the inhomogeneities:

$$h_{ij}=(\sigma e^{\alpha})^2\left[{}^0h_{ij}+2\epsilon(2\pi)^{3/2}\sum\left\{a_{\vec{n},\pm}(t)Q_{\vec{n},\pm}\,{}^0h_{ij}+b_{\vec{n},\pm}(t)\left(\frac{3}{\omega_n^2}(Q_{\vec{n};\pm})_{,ij}+Q_{\vec{n},\pm}\,{}^0h_{ij}\right)\right\}\right],$$

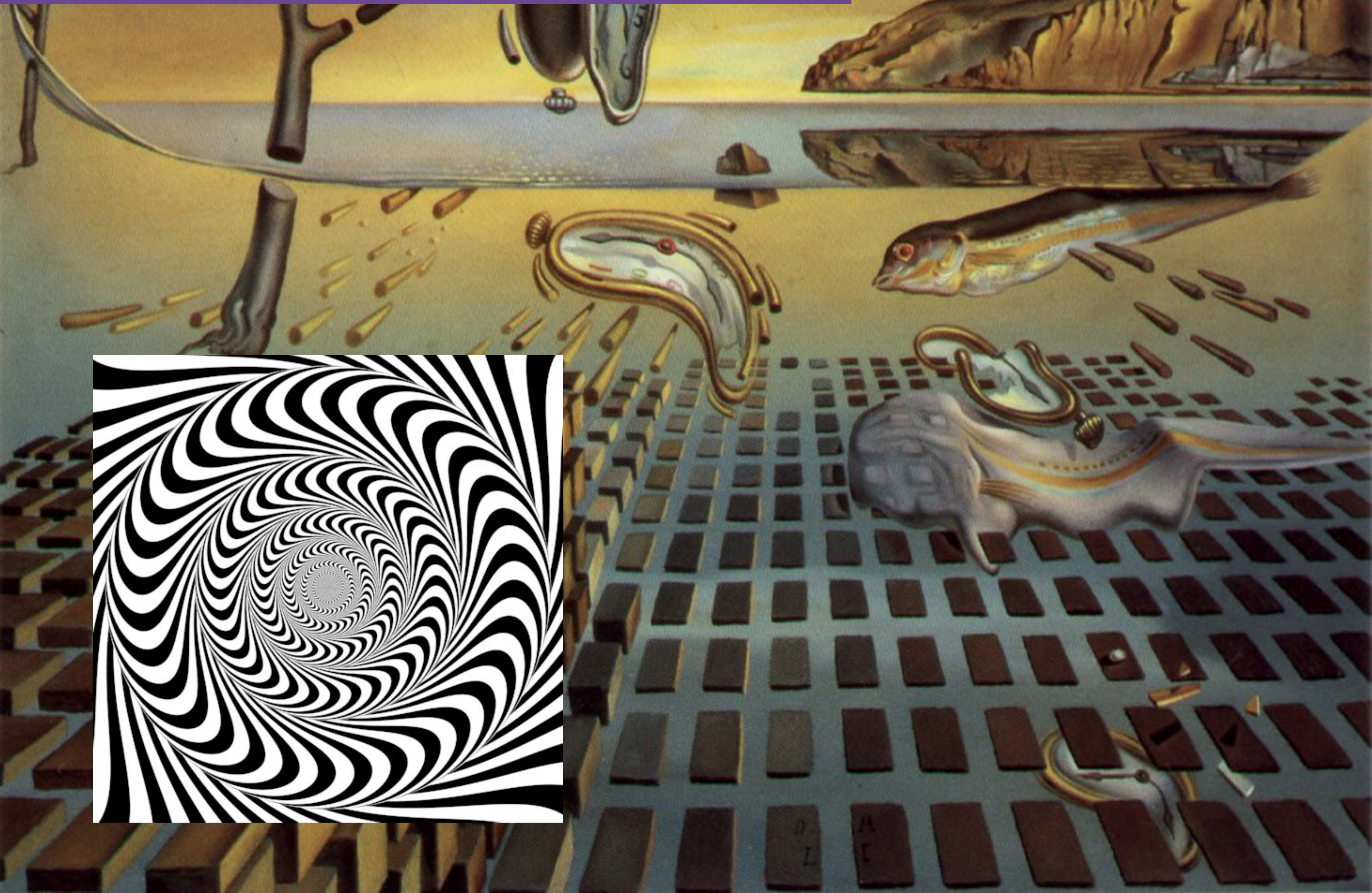
$$N=\sigma N_0(t)\left[1+\epsilon(2\pi)^{3/2}\sum g_{\vec{n},\pm}(t)Q_{\vec{n},\pm}\right],\qquad N_i=\epsilon(2\pi)^{3/2}\sigma^2 e^{\alpha}\sum\frac{k_{\vec{n},\pm}(t)}{\omega_n}(Q_{\vec{n},\pm})_{;i},$$

$$\Phi=\frac{1}{\sigma}\left[\frac{\varphi(t)}{(2\pi)^{3/2}}+\epsilon\sum f_{\vec{n},\pm}(t)Q_{\vec{n},\pm}\right].$$

- Truncating at **quadratic** order in perturbations:

$$H=\frac{N_0\sigma}{16\pi G}C_0+\epsilon^2\sum\left(N_0H_2^{\vec{n},\pm}+N_0g_{\vec{n},\pm}H_1^{\vec{n},\pm}+k_{\vec{n},\pm}\tilde{H}_1^{\vec{n},\pm}\right).$$

# Longitudinal gauge: Hamiltonian



# Longitudinal gauge: Reduction

- After **REDUCTION**, a canonical set is:

$$\bar{\varphi}=\varphi+\epsilon^2 3\sum a_{\vec{n},\pm} f_{\vec{n},\pm}, \qquad \pi_{\bar{\varphi}}=\pi_{\varphi},$$

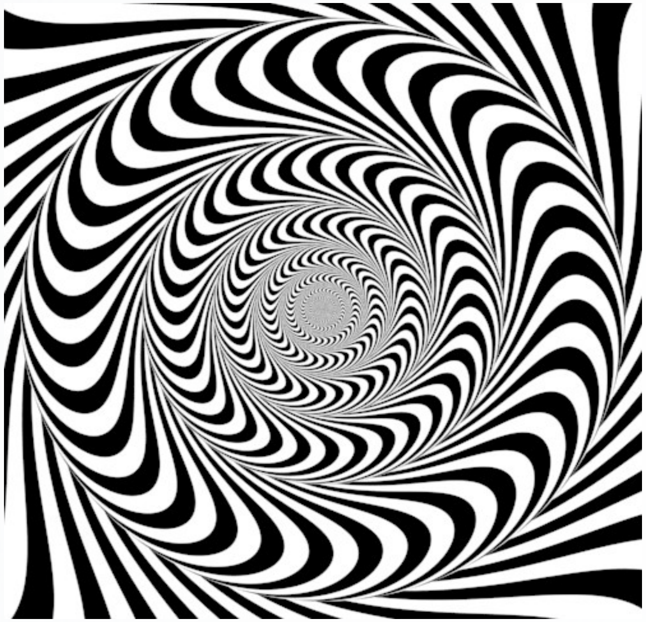
$$\bar{\alpha}=\alpha+\frac{\epsilon^2}{2}\sum\left(a^2_{\vec{n},\pm}+f^2_{\vec{n},\pm}\right), \qquad \pi_{\bar{\alpha}}=\pi_{\alpha}-\epsilon^2\sum f_{\vec{n},\pm}\left(\pi_{f_{\vec{n},\pm}}-3\pi_{\varphi}a_{\vec{n},\pm}-\pi_{\alpha}f_{\vec{n},\pm}\right),$$

$$\bar{f}_{\vec{n},\pm}=e^{\alpha}f_{\vec{n},\pm}, \qquad \pi_{\bar{f}_{\vec{n},\pm}}=e^{-\alpha}\left(\pi_{f_{\vec{n},\pm}}-3\pi_{\varphi}a_{\vec{n},\pm}-\pi_{\alpha}f_{\vec{n},\pm}\right).$$

The genuine background variables are corrected with **quadratic** perturbations.

We have already **scaled** the matter field variables.

# Longitudinal gauge: Dynamics



- The modes of the scaled matter field satisfy a quasi-KG equation with time-dependent mass:

$$\ddot{\bar f}_{\vec n,\pm} + r_n \dot{\bar f}_{\vec n,\pm} + (\omega_n^2 + s + s_n)\bar f_{\vec n,\pm} = 0,$$
$$\pi_{\bar f_{\vec n,\pm}} = (1+p_n)\dot{\bar f}_{\vec n,\pm} + q_n \bar f_{\vec n,\pm},$$

$$s = m^2\sigma^2 e^{2\bar\alpha} - \frac{e^{-4\bar\alpha}}{2}\left(\pi_{\bar\alpha}^2 + 21\pi_{\bar\varphi}^2 + 3e^{6\bar\alpha}m^2\sigma^2\bar\varphi^2\right).$$

$r_n,\ s_n,\ p_n,\ q_n$ are of order $\omega_n^{-2}$.

- For any given background, there exists a **UNIQUE** Fock quantization with the symmetry of the three-torus and unitary dynamics.

- The system can be put in the form of a KG field with time-dependent mass by means of a **mode-dependent** canonical quantization, varying in time.

- This transformation is **unitarily** implementable in the privileged quantization.

# Longitudinal gauge: Hamiltonian

- The remaining **Hamiltonian constraint** reads:

$$H=\frac{N_0\sigma}{16\pi G}C_0+\epsilon^2 N_0\sum H_2^{\vec{n},\pm}, \qquad H_2^{\vec{n},\pm}2e^{\bar{\alpha}}=\bar{E}_{\bar{f}\bar{f}}\bar{f}_{\vec{n},\pm}^2+\bar{E}_{\bar{f}\pi}\bar{f}_{\vec{n},\pm}\pi_{\bar{f}_{\vec{n},\pm}}+\bar{E}_{\pi\pi}\pi_{\bar{f}_{\vec{n},\pm}}^2,$$

$$\bar{E}_{\bar{f}\bar{f}}^n=\omega_n^2+e^{2\bar{\alpha}}m^2\sigma^2-\frac{e^{-4\bar{\alpha}}}{2}\left(\pi_{\bar{\alpha}}^2+15\pi_{\bar{\varphi}}^2+3e^{6\bar{\alpha}}m^2\sigma^2\bar{\varphi}^2\right)-\frac{3}{\omega_n^2}e^{-8\bar{\alpha}}\left(e^{6\bar{\alpha}}m^2\sigma^2\bar{\varphi}-2\pi_{\bar{\alpha}}\pi_{\bar{\varphi}}\right)^2.$$

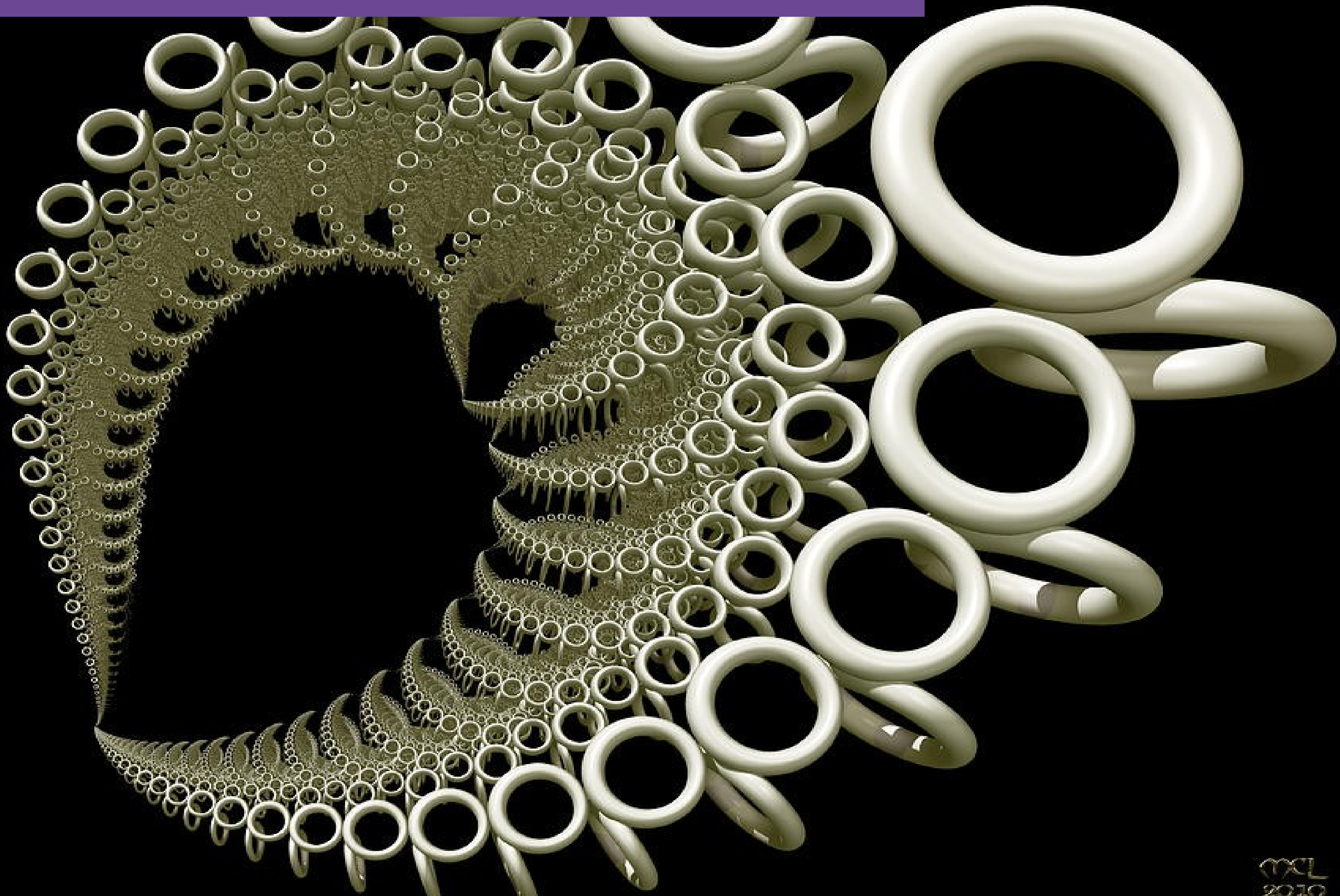
$$\bar{E}_{\bar{f}\pi}^n=-\frac{3}{\omega_n^2}e^{-6\bar{\alpha}}\pi_{\bar{\varphi}}\left(e^{6\bar{\alpha}}m^2\sigma^2\bar{\varphi}-2\pi_{\bar{\alpha}}\pi_{\bar{\varphi}}\right), \qquad \bar{E}_{\pi\pi}^n=1-\frac{3}{\omega_n^2}e^{-4\bar{\alpha}}\pi_{\bar{\varphi}}^{2.}$$

The corrections in cyan are of order $\omega_n^{-2}$.

# Robustness under gauge fixing

- A unitary transformation relates the reduced variables for the inhomogeneities with the **Mukhanov-Sasaki** variables.
- Similar results are obtained in the gauge of flat spatial sections.
- Moreover, the same **symplectic** structure for **gauge invariants** is obtained.

# Quantization: Homogeneous sector



# Quantization: Homogeneous sector

- We quantize the homogeneous sector with standard loop techniques.
- We can adopt a basis of volume eigenstates $\{|v\rangle;\, v\in\mathbb{R}\}$, with $\hat{V}=|\hat{p}|^{3/2}$.

  The inner product is **discrete**: $\forall\, v_1, v_2\in\mathbb{R},\ \langle v_1|v_2\rangle=\delta^{v_1}_{v_2}$.

- The volume and triad act by multiplication.
- It suffices to consider holonomies along (fiducial) straight edges:

$$h_{{}^0 e_i}(\bar{\mu})=\cos\left(\frac{\bar{\mu}\,c}{2}\right)\mathbf{1}+2\sin\left(\frac{\bar{\mu}\,c}{2}\right)\tau_i .$$

- The holonomies elements are linear combinations of $N_{\bar{\mu}}:=e^{i\bar{\mu}c/2}$.

# Quantization: Homogeneous sector

- We use the so-called improved dynamics and the MMO proposal.
- In the volume basis:

$$\hat{N}_{\bar{\mu}}|v\rangle := |v+1\rangle, \quad \hat{p}|v\rangle = sgn(v)\left(2\pi\gamma G\hbar\sqrt{\Delta}|v|\right)^{2/3}|v\rangle.$$

- The kinematic Hilbert space is $H_{kin}^{FRW-LQC} \otimes H_{kin}^{matt}$.
- The inverse volume is regularized as usual in LQC.

$$\widehat{\left[\frac{1}{V}\right]} = \widehat{\left[\frac{1}{\sqrt{|p|}}\right]}^{3}, \quad \widehat{\left[\frac{1}{\sqrt{|p|}}\right]} = \frac{3}{4\pi\gamma G\hbar\sqrt{\Delta}}\widehat{sgn(p)}\sqrt{|\hat{p}|}\left(\hat{N}_{-\bar{\mu}}\sqrt{|\hat{p}|}\hat{N}_{\bar{\mu}} - \hat{N}_{\bar{\mu}}\sqrt{|\hat{p}|}\hat{N}_{-\bar{\mu}}\right).$$

# Quantization: Homogeneous Hamiltonian

- After **decoupling the zero-volume** state, we change the densitization in the FRW constraint:

$$\hat{C}_0 = \left[\widehat{\frac{1}{V}}\right]^{1/2} \hat{\mathcal{C}}_0 \left[\widehat{\frac{1}{V}}\right]^{1/2}.$$

$$\hat{\mathcal{C}}_0 = -\frac{6}{\gamma^2}\hat{\Omega}_0^2 + 8\pi G\left(\hat{\pi}_\phi^2 + m^2\hat{\phi}^2\hat{V}^2\right).$$

- The gravitational part, with the **MMO proposal**, is:

$$\hat{\Omega}_0 = \frac{1}{4i\sqrt{\Delta}}\hat{V}^{1/2}\left[\widehat{sgn(p)}\left(\hat{N}_{2\bar{\mu}} - \hat{N}_{-2\bar{\mu}}\right) + \left(\hat{N}_{2\bar{\mu}} - \hat{N}_{-2\bar{\mu}}\right)\widehat{sgn(p)}\right]\hat{V}^{1/2}.$$

Takes into account the triad orientation (manifest in anisotropic scenarios).

- This operator has the generic form

$$\widehat{\Omega}_0^2|v\rangle = f_+(v)|v+4\rangle + f(v)|v\rangle + f_-(v)|v-4\rangle.$$

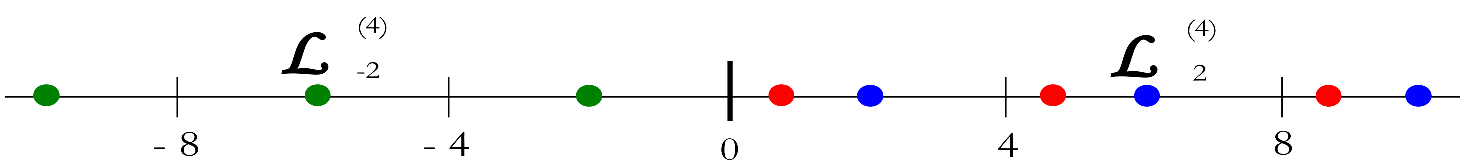
# Quantization: Superselection

- $\hat{\Omega}_0^2$ can be seen as a difference operator.

$$\hat{\Omega}_0^2|v\rangle = f_+(v)|v+4\rangle + f(v)|v\rangle + f_-(v)|v-4\rangle.$$

- The real function $f_+(v)$ $\left(f_-(v)\right)$ vanishes in the **interval** $[-4,0]$ $\left([0,4]\right)$.

- The operator preserves the **superselection** sectors $\mathscr{L}^{(4)}_{\pm\epsilon} := \{\pm(\epsilon+4n),\ n\in\mathbb{N}\}$

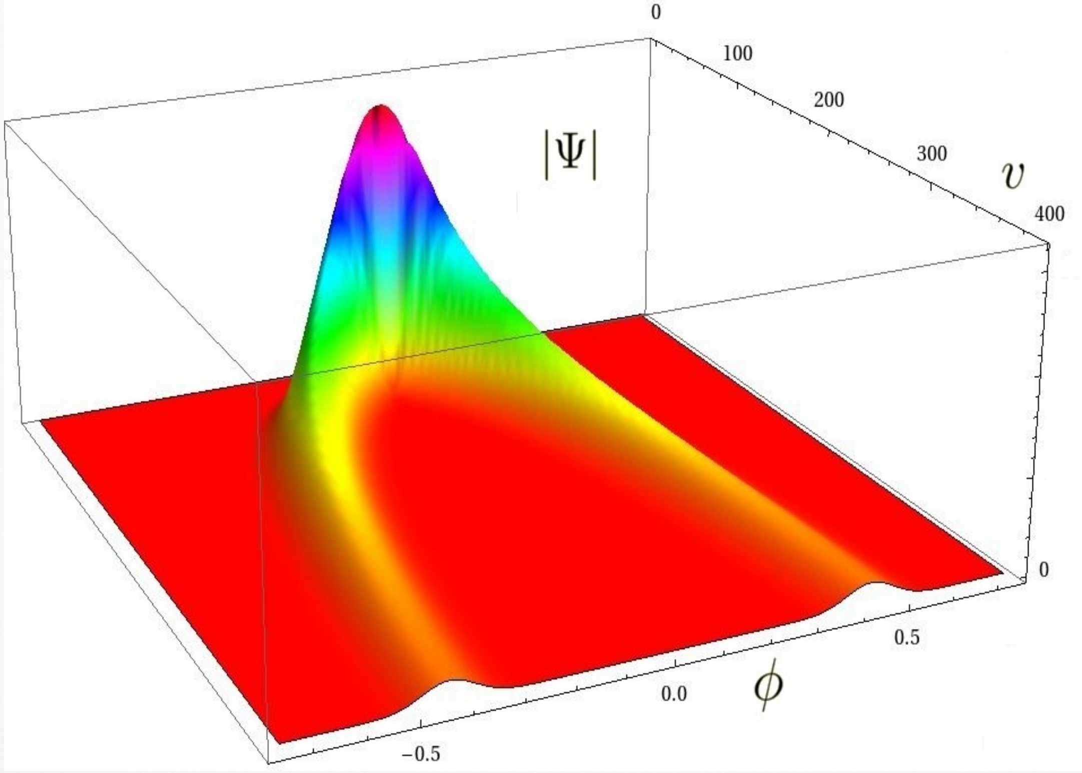



- This operator is selfadjoint in those sectors. Its eigenfunctions are real, and determined by their value at the **minimum volume** $\epsilon\in(0,4]$.

# Quantization: Homogeneous states

- **Solutions** to the constraint are determined, e.g., by their initial values at minimum volume.

- If the scalar field serves as a clock, an alternate possibility is to give the value at a section of constant field.




The space of *physical* states can be identified, e. g., with

$$L^2(\mathbb{R}, d\,\phi).$$

# Fock and hybrid quantizations

- We quantize the **rescaled inhomogeneous modes** using annihilation and creation variables constructed from our canonical variables and zero mass.

- We obtain a **Fock space** $\mathscr{F}$ , with basis of n-*particle* states:

$$\left\{ |N\rangle = |N_{(1,0,0),+}, N_{(1,0,0),-}, \ldots\rangle; \quad N_{\vec{n},\pm} \in \mathbb{N}, \ \sum N_{\vec{n},\pm} < \infty \right\}.$$

- We proceed to a hybrid quantization, with Hilbert space

$$H_{kin}^{FRW-LQC} \otimes H_{kin}^{matt} \otimes \mathscr{F} .$$

- The Hamiltonian constraint is **not trivial.**

# Quantum Hamiltonian of the perturbations

- We quantize the quadratic contribution of the perturbations to the Hamiltonian adapting the quantization **proposals of the homogeneous sector** and using a symmetric factor ordering:

  - We **symmetrize** products of the type $\hat{\phi}\,\hat{\pi}_{\phi}$.
  - We take a **symmetric geometric** factor ordering $V^k A \rightarrow \hat{V}^{k/2} \hat{A}\, \hat{V}^{k/2}$.
  - We adopt the **LQC** representation $(cp)^{2m} \rightarrow \left[\hat{\Omega}_0^2\right]^m$.
  - In order to **preserve the FRW superselection sectors**, we adopt the prescription $(cp)^{2m+1} \rightarrow \left[\hat{\Omega}_0^2\right]^{m/2} \hat{\Lambda}_0 \left[\hat{\Omega}_0^2\right]^{m/2}$, where

$$\hat{\Lambda}_0 = -\frac{i}{8\sqrt{\Delta}}\,\hat{V}^{1/2}\left[\widehat{sgn(p)}\left(\hat{N}_{4\bar{\mu}} - \hat{N}_{-4\bar{\mu}}\right) + \left(\hat{N}_{4\bar{\mu}} - \hat{N}_{-4\bar{\mu}}\right)\widehat{sgn(p)}\right]\hat{V}^{1/2}.$$

The situation is similar to that found with the Hubble parameter in LQC.

# Solutions to the constraint

- If the matter field serves as a **clock**, we can:

  - ☆ Consider positive (negative) frequency states with respect to to that time.
  - ☆ Use a Born-Oppenheimer-like **approximation** $\Psi=\chi_0(v,\phi)\,\psi(\phi,N[\bar{f}_{\vec{n},\pm}])$.
  - ☆ **Neglect** the field momentum of the inhomogeneities versus that of the homogeneous part.

- This leads to a sort of ***effective*** QFT for the inhomogeneities.

$$-i\hbar\partial_\phi\tilde{\psi}=\frac{\epsilon^2}{2}\frac{\langle{}^{(0)}\hat{\Theta}_2+{}^{(1)}\hat{\Theta}_2\hat{H}_0\rangle_{\chi_0}}{\langle\hat{H}_0\rangle_{\chi_0}}\tilde{\psi}.$$

$$\hat{H}_0^2={}^{(0)}\hat{\pi}_\phi^2-\frac{\hat{C}_0}{8\pi G},\qquad \tilde{\psi}=\langle\hat{H}_0\rangle_{\chi_0}\psi,$$

$$\hat{H}_2^{\vec{n},\pm}=\frac{\sigma}{16\pi G}\widehat{\left[\frac{1}{V}\right]}^{1/2}\hat{C}_2^{\vec{n},\pm}\widehat{\left[\frac{1}{V}\right]}^{1/2},$$

$$\sum\hat{C}_2^{\vec{n},\pm}=-8\pi G\left({}^{(0)}\hat{\Theta}_2-i\hbar\,{}^{(1)}\hat{\Theta}_2\partial_\phi\right).$$

# Physical states

- An alternate **perturbative** scheme:

$$(\Psi| = (\Psi|^{(0)} + \epsilon^2 (\Psi|^{(2)} \ldots$$

- **FRW solution:** $(\Psi|^{(0)} \hat{C}_0 = 0,$

$$\hat{C}_0 = -\frac{6}{\gamma^2}\hat{\Omega}_0^2 + 8\pi G\left(\hat{\pi}_\phi^2 + m^2 \hat{\phi}^2 \hat{V}^2\right).$$

- ***Evolution* of the perturbations:**

$$(\Psi|^{(2)} \hat{C}_0 = -(\Psi|^{(0)} \left(\sum \hat{C}_2^{\vec{n},\pm}\right)^\dagger.$$

- Solutions are characterized by their initial data at **minimum volume.**
- From these data we arrive, e.g., at the **physical Hilbert space** $H_{kin}^{matt} \otimes \mathscr{F}$ .

# Conclusions

- We have considered a perturbed FRW universe with a **massive** scalar field.
  - ✯ The action has been truncated to second order in the perturbations.
  - ✯ A hybrid quantization scheme has been adopted.
- The system is endowed with a **symplectic structure** and a **Hamiltonian** constraint. **Backreaction** is included.
- **No internal time** is needed.
- Under some **controlled approximations**, if a matter clock is available, one may reach an effective QFT with first-order evolution equations.
- In our analysis, the dynamics are **UNITARY** in the QFT regime.
- One can characterize quantum states from data at **minimum volume**.